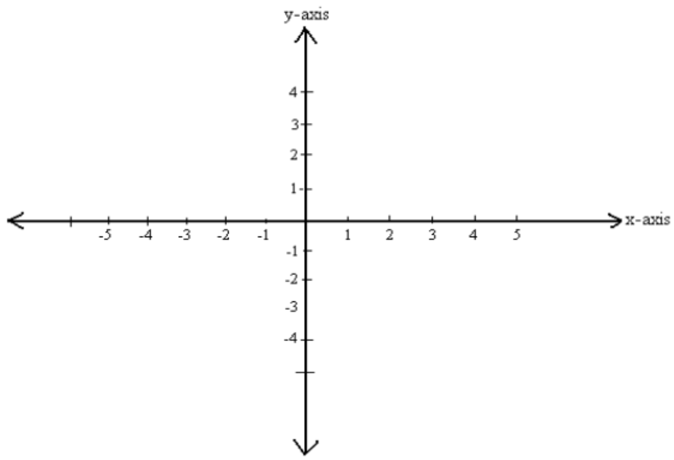
Caltech Math Circle
Seventh grade group
December 5, 2023

Linear transformations in the plane

Loosely, a linear transformation is a function from the plane $\mathbb{R}^2$ to the plane $\mathbb{R}^2$ that sends lines through the origin to lines through the origin or points.

Problem 1. Below is two lines that form the coordinate axes. Determine ways in which you can transform the axes so that every line through the origin in the plane ends up as a line through the origin or a point.



Problem 2. Let f send a point (x, y) in the plane to (x^2, y^2). Where does f send the line $y = x$? Is f a linear transformation?

Problem 3. Let R be rotation by $90°$ counterclockwise. Let T be reflection about the x-axis. Is applying T then R the same as applying R then T?

Definition 1. Let T be a linear transformation. An *inverse* transformation S of T is a one for which applying S then T or T then S keeps every point in the plane in the same place.

Problem 4. Describe an inverse transformation for reflection about the x-axis.

Problem 5. Describe an inverse transformation for rotation by $90°$ counterclockwise.

Problem 6. Is there an inverse for projection onto the x-axis?

Problem 7. Determine lines that remain unchanged by each of the following transformations.

(a) Scaling by a factor of 2
(b) Rotation by $90°$ counterclockwise.
(c) Reflection about the x-axis.
(d) Projection onto the x-axis

Can we classify the linear transformation by the lines that each one fixes?

Matrix representation of linear transformations

Definition 2. For our purposes, a *vector* is an ordered pair of numbers, $\begin{pmatrix} x \\ y \end{pmatrix}$, the first of which represents the x-coordinate and the second of which represents the y-coordinate in the Cartesian plane.

Problem 8. Let $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be two vectors.

(a) Determine where we send i and j when we rotate the plane by 90° counterclockwise.
(b) Determine where we send i and j when we reflect about the x-axis.
(c) Determine where we send i and j when project onto the x-axis.

Definition 3. A *matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an array of numbers that can be used to represent linear transformations. The matrix has two rows: $\begin{pmatrix} a & b \end{pmatrix}$ and $\begin{pmatrix} c & d \end{pmatrix}$ and two columns: $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$.

Problem 9. The first column of the matrix representation of a linear transformation is where we send $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ via the transformation. The second column of the matrix representation is where we send $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ via the transformation.

(a) Find the matrix representation of rotation by 90° counterclockwise.
(b) Find the matrix representation of the transformation that reflects about the x-axis.
(c) Find the matrix representation of the transformation that projects onto the x-axis.

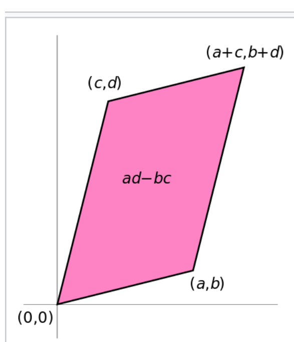
With the above construction, applying a transformation to a vector can be done by multiplying the vector on the left by its matrix representation.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Problem 10. Determine where each transformation from Problem 9 sends the general vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by multiplying on the left by the corresponding matrix.

Definition 4. The *determinant* of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det(A) = ad - bc$.

The determinant measures the area of the parallelogram formed by the first and second columns of the matrix. In other words, it measures how much the linear transformation scales area.



Problem 11. Compute the determinant of the matrix representations in Problem 9. Does the determinant correspond to how the transformations affect area in the plane?

Problem 12. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Which linear transformation in the plane does A represent?

(b) Apply the transformation to a general vector $\begin{pmatrix} x \\ y \end{pmatrix}$

(c) How does A scale the area of the unit square in the first quadrant?

Definition 5. The *trace* of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the sum of its diagonal entries, $\operatorname{tr}(A) = a + d$.

Problem 13. Write down two 2×2 matrices A and B for which $\det(A) = \det(B)$ and $\operatorname{tr}(A) = \operatorname{tr}(B)$ but A is not equal to B.

Composition of transformations

Problem 14. (a) Find a matrix representation for the linear transformation in which we first rotate by 90° counterclockwise and then reflect about the x-axis.

(b) Find a matrix representation for the linear transformation in which we first reflect about the x-axis and then rotate by 90° counterclockwise.

(c) Are the transformations from (a) and (b) the same?

Recall the matrices from Problem 9(a) and (b) are $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively. The first matrix represents rotation by 90° counterclockwise. The second matrix represents reflection about the x-axis. We can multiply the matrices as follows.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} (0)(1) + (-1)(0) & (0(0) + (-1)(-1) \\ (1)(1) + (0)(0) & (1)(0) + (0)(-1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The ith column of the product is the first matrix times the ith column of the second matrix.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (0)(1) + (-1)(0) \\ (1)(1) + (0)(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} (0)(0) + (-1)(-1) \\ (1)(0) + (0)(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Problem 15. (a) Does the matrix represent rotation then reflection or reflection then rotation?

(b) Multiply the matrices in the other order to obtain the other matrix from Problem 14.

(c) Compute the determinant of the product of matrices in either order. How does the composition affect areas in the plane?

Problem 16. Multiply the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in both orders. Explain what linear transformation each matrix represents.

Problem 17. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

(a) Determine the matrix product AB.

(b) Compute $\det(A)$ and $\det(B)$.

(c) Find a formula for $\det(AB)$ in terms of $\det(A)$ and $\det(B)$.

(d) Determine $\det(BA)$ without multiplying the matrices B and A.

Problem 18. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

(a) Compute $\operatorname{tr}(A)$ and $\operatorname{tr}(B)$.

(b) Is there a formula for $\operatorname{tr}(AB)$ in terms of $\operatorname{tr}(A)$ and $\operatorname{tr}(B)$?